Running Head: Combinatorics Problems Represented Using Arrays

An Investigation of 6th Graders' Solutions of Cartesian Product Problems and Representation of these Problems Using Arrays

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Abstract

Two hour-long interviews were conducted with each of 14 sixth-grade students. The purpose of the interviews was to investigate how students solved combinatorics problems, and represented their solutions as arrays. This paper reports on 11 of these students who represented a balanced mix of students operating with two of three multiplicative concepts that have been identified in prior research (Hackenberg & Tillema, 2009; Hackenberg, 2007, 2010). One finding of the study was that students operating with different multiplicative concepts established and structured pairs differently. A second finding is that these different ways of operating had implications for how students produced and used arrays. Overall, the findings contribute to models of students' reasoning that outline the psychological operations that students use to constitute product of measures problems (Vergnaud, 1983). Product of measures problems are a kind of multiplicative problem that has unique mathematical properties, but researchers have not yet identified specific psychological operations that students use when solving these problems that differ from their solution of other kinds of multiplicative problems (cf. Battista, 2007).

1. Introduction

Researchers have identified that instruction that supports students to represent sets of outcomes in their solution of combinatorics problems consistently yields positive results (e.g., English, 1991; Maher, Powell, & Uptegrove, 2010). For example, Fischbein and colleagues (Fischbein, Pampu, & Minzat, 1970; Fischbein & Gazit, 1988) found that the use of tree diagrams increased the number of problems middle grades students (ages 11-14) successfully solved, while Lockwood (2014) found that listing sets of outcomes helped college students correct and explain common counting errors. In curricular materials, one common way to represent sets of outcomes is with two-dimensional arrays (e.g., Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002; Van de Walle, Karp, & Bay-Williams, 2011). Arrays match well with unique aspects of the multiplicative reasoning involved in the solution of combinatorics problems (Behr, Post, Lesh & Harel, 1994; Vergnaud, 1983). However, relatively few empirical studies have investigated how students produce such representations in combinatorial contexts (see Outhred, 1996 for one study). More broadly, Battista (2007) has identified a dearth of research that provides models of how students *relate* one and two-dimensional units to each other, although his focus in making this claim was on geometric not combinatorial contexts. Such research is critical given the widespread use of two-dimensional arrays to represent mathematical situations.

The purpose of this paper is to investigate: 1) the mental operations 11 6^{th} grade students used in their solution of Cartesian product problems like the Outfits Problem;

> *Outfits Problem:* You have four shirts and three pairs of pants. An outfit consists of one shirt and one pants. How many possible outfits could you make?

and 2) how these operations are connected to the ways in which they represented sets of outcomes as two-dimensional arrays. The data are drawn from an interview study with a total of

14 6th grade students. This paper focuses on 11 students who were using the first two of three multiplicative concepts that have been identified in prior research (Hackenberg 2007, 2010; Hackenberg & Tillema, 2009). Students using the first multiplicative concept (MC1 students) can establish a unit of units in activity. Heuristically, this mean that to solve an equal groups multiplication problem they track the number of groups and the number in a group (two levels of units) as part of their activity. Students using the second multiplicative concept (MC2 students) can take a unit of units as a given. Heuristically, this means that tracking the number of groups and the number in a group is no longer a central part of their activity, which allows them to reason strategically with composite units (Steffe, 1992; Ulrich, 2015). The reason that this paper *only* focuses on MC1 and MC2 students' solutions of Cartesian product problems is that an earlier study (Tillema, 2013) focused on how students using the third multiplicative concept (MC3 students) solved Cartesian product problems. Thus, this paper extends the earlier study by investigating a broader range of students' reasoning on Cartesian product problems. The following research questions guide the study:

(RQ1) What mental operations did students use to solve Cartesian product problems?

(RQ2) What differences (if any) were there in how MC1 and MC2 students established multiplicative relationships in their solution of Cartesian product problems?

(RQ3) How did these differences impact how they developed two-dimensional arrays?

2. Literature Review

To understand how Cartesian product problems can support students to relate one and two-dimensional units, I draw on researchers' theoretical analyses of the differences and similarities between Cartesian product problems and other problem situations that can involve multiplication. I then provide a review of empirical studies to illustrate that researchers' models of students' reasoning have not yet captured how Cartesian product problems could support students to relate one and two-dimensional units.

2.1. Theoretical Analyses of Multiplicative Reasoning, Attention to Unit Structures, and Arrays

In providing a mathematical characterization of the multiplicative conceptual field, Vergnaud (1983) differentiated between isomorphism of measures problems like the Donut Problem and product of measures problems like the Outfits or Area Problem (see also, Behr, Post, Lesh & Harel, 1994; cf. Greer, 1992).

Donut Problem: There are three packages of donuts. There are four donuts in each package. How many total donuts are there? (isomorphism of measures problem)

Area Problem: A rectangle has a width of four units and a length of three units. What is its area? (see Introduction for the Outfits Problem; both are product of measures problems)

Vergnaud noted that in product of measures problems, the unit that is enumerated can be constituted from the product one times one. For example, in the Outfits Problem one shirt paired with one pants is equal to one outfit (1 x 1 is 1). Elsewhere, Tillema (2013, 2016) has called units like outfits *pairs* because they result from a multiplicative composition of two more basic units, but they are counted as a single unit (Outhred, 1996; Vergnaud; Behr, et. al.).

Behr et. al. (1994) followed up on Vergnaud's work with a theoretical analysis of how the unit structures in product of measures problems could differ from the unit structures in isomorphism of measure problems. To establish a composite unit in the Donut Problem, a student can make a one-to-many coordination between the first package and four donuts—one package contains four donuts (Figure 1a). To establish a composite unit in the Outfits Problem, a student first needs to pair the first shirt with each pants before making a one to many coordination between the first pants and the four outfits containing the first pants (Figure 1b).

The primary difference between Figure 1a and 1b is that each "donut" contained in a composite unit is a unit of one (called a unit of four units), whereas the "outfits" are not—outfits are already constituted from two units of one (called a unit of four pairs).

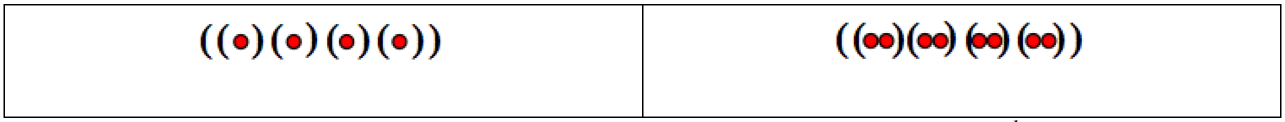

Figure 1a (left) & 1b (right): Difference in unit structures[1]

Establishing pairs is not dissimilar from establishing area units—a student can constitute area units as the product of two more elementary units, a length and width unit. When considering students' solution of area problems, central issues are whether, when, and how students multiplicatively compose length and width units to create area units, and what relationships they establish among length, width, and area units (Battista, 2007). For example, do students see a row in a rectangular figure as a product of one length unit with four width units that creates four area units? Or, do they simply see the row as four units that have *not* been constituted from the product of length and width units (Simon & Blume, 1994; Thompson, 2000)? In the former case, a student might establish a row as a multiplicative relationship among one length unit, a unit of four width units, and a unit of four area units, whereas in the latter case a student might only establish the row as a unit of four units. In the latter case, I would not use the term "area units" (i.e., a unit of four area units) because the four units were not constituted as a product of length and width units. In this case, from the perspective of unit structure, creating one row containing four units is similar to creating one package containing four donuts, and so similar language (i.e., a unit of four units) is used to describe the unit structure.

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[1] In the figures I represent each additional level of interiorized unit with a parenthesis. I use a dashed parenthesis when a student can establish a particular unit structure in activity.

Although combinatorics problems are different from area problems in that they involve discrete rather than continuous units, these issues are still important for understanding how students solve combinatorics problems and how they represent them as two-dimensional arrays. For example, a student might establish a multiplicative relationship among a unit of one (one pants), a unit of four units (four shirts), and a unit of four pairs (four outfits) (Figure 2). Figure 2 differs from Figure 1b in that Figure 2 shows not just the pairs, but is also intended to convey that a student has established and retained a multiplicative relationship among one pants, four shirts, and four outfits. Establishing the kind of relationship shown in Figure 2 has the potential to undergird seeing a row in an array as a multiplicative relationship among a unit represented on one axis, four units represented on the other axis, and four points in the interior of the array where the points represent *pairs*.

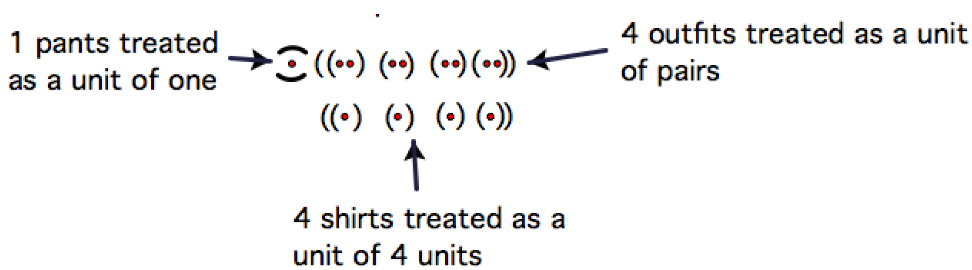

Figure 2. A multiplicative relationship among a unit of one, a unit of four units, and a unit of four pairs.

2.2. Empirical Studies about Multiplicative Reasoning, Attention to Unit Structures, and Arrays

Few empirical analyses of students' reasoning have identified distinct mental operations (i.e., psychological distinctions) that students use to solve product of measures problems (e.g., Outfits or Area problems) versus isomorphism of measures problems (e.g., Donut Problem) (e.g., English, 1991, 1993; Maher & Martino, 1996; Nunes, Light, & Mason, 1993; Nunes & Bryant, 1996). For example, in his teaching experiments Steffe (1992, 1994) asked 3rd grade students to solve both product and isomorphism of measures problems. His analyses accounted for students producing a unit structure that match with Behr and colleagues unit structures for isomorphism of measures problems; that is, Steffe claimed that students might establish a unit of four units (Figure 1a) in the context of solving, for example, Cartesian product problems, rather than a unit of four pairs (Figure 1b). Similarly, Mulligan and Mitchelmore (1997) identified three different intuitive models (direct counting, repeated addition, and multiplicative operation) that students used to solve both isomorphism and product of measures problems. They explained differences in students' use of different intuitive models based on a complex interaction of factors related to age, size of numbers in the problem, language, and semantic structure of the problems. They, too, did not propose that students used any novel mental operations when solving problems that could be classified as product of measures problems.

Outhred (1996) has investigated the relationship between students' solutions of combinatorics problems and their drawn representations of the outcomes as arrays. She found that 4th grade students who used "count all" or "repeated addition" strategies to solve these problems did not use array-like representations, whereas all but one student who multiplied to solve these problems did use array-like representations. Her analysis linked the kind of drawn representation a student used to the way that a student enumerated the number of outcomes (e.g., count all, repeated addition, or multiplication). However, she did not provide an explicit model of whether the unit structure a student produced was more advanced in these problems relative to the unit structure they produced to solve, for example, isomorphism of measures problems.

Other research studies in which this issue has been acknowledged have largely been studies investigating students' solutions of area problems. In a literature review, Battista (2007) noted that numerous studies indicate that students, across a range of ages, may not actually constitute area units as the product of linear units in their solution of area problems (e.g., Outhred & Mitchelmore, 2000). For example, Nunes, Light, and Mason (1993) found that elementary grades students preferred a tiling approach to area measurement (e.g., three rows of square tiles with four square tiles in each row) over an approach where they were asked to relate length (e.g., three cm) and width measurements (e.g., four cm) to area measurement (12 square cm) (see also Mulligan & Mitchelmore, 2000; Thompson, 2000). Similarly, Simon & Blume (1994) found that this issue was difficult for pre-service elementary teachers who used a 3 by 5 index card to measure the area of a rectangular table; many pre-service elementary teachers counted, added, or multiplied to determine the number of index cards that covered the rectangular table, treating the index card as the basic unit of measure rather than constituting an area unit as a product of a length and width unit. These findings have broad implications about how students understand the relationship between linear and area measurement units, which prompted Battista (2007) to call for studies that explicitly analyze how students constitute this relationship. Although combinatorics problems involve discrete units, it is similarly important to understand whether and when students are using mental operations that support them to establish relationships among one and two-dimensional units.

3. Theoretical Framework

3.1. Operations, schemes, and concepts

I characterize students' mathematical activity in terms of operations, which are mental actions. Operations are carried out on figurative material that can be either perceptually available or mentally generated (Piaget, 1970; von Glasersfeld, 1995). For example, to solve the Outfits Problem a student might write on a piece of paper the letters "A", "B", and "C" to represent three pants, and the numbers "1", "2", "3", and "4" to represent four shirts. They could subsequently write a list to represent the outfits, “A1”, “A2”, etc. The student’s establishment of outfits, “A1”, “A2”, etc. can entail a pairing operation, which is the mental action of putting together the letter “A” with the number “1” to create an outfit. In this example, a student carried out an operation on perceptually available figurative material; the letters and numbers are the perceptually available figurative material because they are recorded on a piece of paper. A student could, however, exclusively imagine the letters and numbers, in which case a student would carry out a pairing operation on mentally generated figurative material. Operations, and the figurative material that they operate on, are the building blocks of schemes.

A scheme has three parts, an assimilatory mechanism, an activity, and a result (Piaget, 1970; von Glasersfeld, 1995). The assimilatory mechanism of a scheme involves a student in establishing an interpretation of a problem situation; such an interpretation can trigger a particular activity, where the activity of a scheme entails a student in using operations; operations transform the student’s initial interpretation of the situation into a result. Following from the example above a student might assimilate the Outfits Problem using two composite units, three and four. The activity of their scheme might then entail pairing the first unit from one composite unit with the first unit of the other composite unit, continuing with this process until they had established all possible pairs. The result of the scheme then would be a plurality of pairs. Once the result of a scheme is available to a person prior to operating in a situation they have established a concept (von Glasersfeld, 1982). A person’s current concepts are what he or she uses to assimilate situations.

3.2. Two of Three Multiplicative Concepts

Building from Steffe’s work (1992, 1994), Hackenberg (2007, 2010) has identified three qualitatively distinct multiplicative concepts based on the number of levels of units that a student is coordinating (see also, Hackenberg & Tillema, 2009; Norton & Wilkins, 2012; Ulrich, 2015; cf. Kamii & Housman, 2000). Students move from one multiplicative concept to the next through the process of interiorization—the re-processing of the result of a scheme so that a student can anticipate the result prior to activity. The process of interiorization entails a significant shift in a student's ways of operating. Thus, the three multiplicative concepts represent stages of learning that may last for two or more years (Steffe, 2007).

To date, the unit structures that Hackenberg (2007, 2014) has identified match with the unit structures that have been outlined in theoretical analyses for isomorphism of measures problems (Figure 1a) not product of measures problems (Figure 1b or 2). I use the Donut Problem (from the Literature Review), which is an isomorphism of measure problem, to outline the mental operations and the material that students using the first and second multiplicative concept (MC1 and MC2 students, respectively) operate on to solve such problems.

A distinguishing characteristic of MC1 students is that they have interiorized a single level of unit and can create two levels of units in activity. Their meaning for number words like four and three is a unit of one *iterated* four or three times, respectively. They consider each of the units in a composite unit, like four, to be equivalent to each other, but not identical; each unit is equivalent because each is a unit of one, but they are not identical because the position of each unit in the sequence of units is different (Ulrich, 2015). To solve a problem like the Donut Problem, an MC1 student can assimilate the situation using two composite units, four and three. The student might then engage in what Steffe (1992) has referred to as a units coordination—insert the units of one composite unit into the units of the other composite unit. One indicator of this insertion is counting as follows: "4—that is one package; 5, 6, 7, 8—that is two packages; 9, 10, 11, 12—that is three packages." Here key criteria for inferring that a student is coordinating two levels of units in activity are that they are able to keep track of the number of packages, the number of donuts in each package, and the total number of donuts that have accumulated. However, tracking all three may not be fully worked out. For example, a student asked to count one more package of donuts after figuring out how many donuts three packages of four donuts is might conflate the number of packages with the number of donuts in a package. This conflation might entail them counting three more on to twelve rather than counting four more on to twelve because they momentarily conflate the number of packages that they have tracked so far (three) with the number of donuts in each package (four). These kind of conflations are precisely why MC1 students are considered to create a unit of units in activity.

A distinguishing characteristic of MC2 students is that they have interiorized two levels of units. They consider each of the units in a composite unit, like four, to be identical; the first unit of a composite unit could be iterated to create the composite unit, but does not need to be because the position of each unit is no longer a salient feature for them (Ulrich, 2015). This means that MC2 students can treat a composite unit like it is a unit of one (e.g., they can iterate a unit of four units in a way that is similar to how MC1 students can iterate a unit of one). So to solve the Donut Problem these students might iterate four three times, and in the process of solving the problem they are likely to use strategic reasoning. For example, a student might reason that four and four is eight, and two more than that is ten and the remaining two is twelve. Reasoning in this way can involve using a disembedding operation—a student disembeds two and two from four. Here a *disembedding* operation enables students to treat a part of a composite unit as independent of the composite unit without mentally destroying it (i.e., two and two are simultaneously a part of, and independent from four).

Some MC2 students engage in a further units coordination as part of their activity: They insert the 3 units of 4 units into a containing unit to create a unit of 3 units of 4 units *in activity*—a three-level-of-unit structure. To solve an extension of the Donuts Problem (e.g., Suppose you get twice as many donuts as you have now) an MC2 student that creates a three level of unit structure in activity may, for example, iterate three fours twice, retaining that each twelve is a unit of three units of four units, combine the two twelves to get twenty-four, and establish that twenty-four must be six fours (a unit of six units of four units). They might, however, make conflations in their reasoning in further extensions of the problem (e.g., Suppose you get twice as many donuts as you have now). To solve this problem, they might iterate twenty-four once to create forty-eight, but then conclude that forty-eight is nine fours rather than twelve fours, conflating the number of iterations (one iteration implies three more fours) with the number of fours in each twenty-four (six fours in each twenty-four). This kind of operating contrasts from MC2 students who do not create a three level of unit structure in activity. MC2 students who do not create a three level of unit structure in activity would likely solve the extension of the Donut Problem by simply continuing to iterate individual composite units of four. When students have interiorized two levels of units but do not provide indication that they operate with three levels of units in activity, I consider them to be *emergent* MC2 students, while I consider students who are able to create a three-levels-of-unit structure in activity to be *elaborated* MC2 students.

The above characterization of students' multiplicative reasoning identifies three mental operations that are central to this reasoning—iteration, units coordination, and disembedding—and provides a framework that highlights the embedded nature of units that students create as they reason about multiplication problems. Moreover, the primary differences among students using different multiplicative concepts are the unit structure that they operate on, not the operations that they use. Regardless of the number of levels of units that a student is coordinating, the lowest level of unit is a unit of one (e.g., a donut), and this unit is not constituted from the product of two more elementary units (e.g., a shirt and pants). Thus, this characterization does not capture the potentially more complex unit structure of Cartesian product problems that is identified in Behr, et. al.'s (1994) theoretical analysis.

3.3. A Characterization of Novel Operations in Cartesian Product Problems

As a result of working with three MC3 students, Tillema (2013) identified two additional mental operations—an ordering and pairing operation—that helped to characterize whether and when students created a more complex unit structure in their solution of combinatorics problems. An ordering operation entailed students in creating, for example, a first shirt, second shirt, etc. in the Outfits Problem. A pairing operation entailed pairing a unit from one composite unit (e.g., a shirt) with a unit from a second composite unit (e.g., a pants) to create a pair (e.g., an outfit), a unit that contained two units, but was counted as a single unit. The use of a pairing operation was central to students producing a more complex unit structure in combinatorics problems—namely they established a pair as a product of two more elementary units. Because the use of a pairing operation can be seen as a psychological root for the multiplicative identity that one times one is one, Tillema (2013) has considered students use of it to be multiplicative in nature even if, for example, they simply count the pairs that they create by one (cf. Schwartz, 1988; Vergnaud, 1983).

The central issues under investigation for this paper, then, are whether and how MC1 and MC2 students used these five operations (iteration, units coordination, disembedding, ordering, and pairing) to solve combinatorics problems (RQ1 and RQ2) and how these operations supported their creation of arrays (RQ3).

4. Methods and Data Sources

4.1 Interview Study Methodology

The goal of clinical interview methodology is to capture students' authentic ways of reasoning in a particular problem domain (Clement, 2000). Like teaching and design experiment methodologies, a researcher using clinical interview methodology tries to harmonize with students' current ways of operating (Cobb & Gravemeijer, 2008; Confrey & LaChance, 2000; Lobato, 2008; Steffe & Thompson, 2000). The process of harmonizing with students' current ways of operating takes place via a researcher anticipating how students might reason on particular problems prior to the actual interviews and making in-the-moment adjustments to students based on the researcher's actual interactions with them. As part of the interactions, the researcher tests out conjectures formulated prior to interacting with the students, as well as conjectures made during the interaction (Steffe & Thompson, 2000). This approach means that researchers have a tightly designed structure for the interview protocol that enables them to test out particular conjectures, but the researcher may also deviate from the interview protocol in order to test out in-the-moment conjectures.

Unlike teaching and design experiment methodologies, clinical interview methodology only provides a snapshot of students' reasoning. Therefore, it is difficult to make claims about students' learning or what students could learn. Instead, clinical interview methodology allows a researcher to get an authentic experience of how students reason in a particular problem domain, which is especially useful when there has been little prior research in the domain (Clement, 2000). Such a methodology was appropriate for the purposes of this study because combinatorics problems have not been widely used to investigate how students establish more complex unit structures and how this process could support them to develop arrays.

4.2 Data Collection and Participants

The research team consisted of four members, a mathematics education researcher, a mathematics education graduate student, and two undergraduate researchers. The team worked one day a week with a 6th grade teacher in her regular classroom over the course of one semester in order to help the 6th grade students become familiar with the researchers. This work included regular interactions with students both one-on-one and in small groups, co-teaching or teaching full classroom lessons, and working with the teacher on planning instruction. The teacher was interested in supporting student thinking, but did *not* consider herself to be an expert in doing so. Thus her instruction often focused on mathematical procedures (rather than discussion of students' thinking or ways of representing that thinking), but she was willing to try suggestions that the research team made to her.

At the end of the semester, students were asked about their interest in participating in the interview study. For those that expressed interest, the research team conducted un-recorded selection interviews in order to assess students' multiplicative concepts (Appendix A). The selection interviews did not involve combinatorics problems because the purpose was to identify the unit structure that students created in problems that did not involve the potentially more complex unit structure that combinatorics problems could entail. Thus the selection interviews involved problems that were intended to allow the research team to determine the number of levels of units that students coordinated when the initial unit was taken to be a unit of one (rather than a pair). Specifically, they involved problems that could involve multiple levels of embedded units (e.g., problem 4, appendix A) and problems that switched between the kind of unit that is referred to in the problem (e.g., problem 1, Appendix A) in order to investigate the kind of units that students were facile reasoning about.

Fourteen students participated in the study—three of these students were using the third multiplicative concept and are *not* reported on here. The other 11 students, who are a focus of this paper were using either the first or second multiplicative concept; three were MC1 students, five were *emergent* MC2 students, and three were elaborated MC2 students. The 11 students participated in two hour-long, semi-structured, video-recorded interviews (Appendix B and C).[2] At least two members of the research team attended all interviews; one member interviewed while other members took notes on the interactions. The researchers attempted to conduct the two interviews for a student within a short time period since the second interview developed ideas that were established in the first. On average the second interview was conducted within seven days of the first interview. The most common (and shortest) number of days between the two interviews was two days, which occurred for six students. The least common (and longest) number of days between interviews was twenty days, which occurred for one student.

The interview protocols began with Cartesian product problems, (e.g., Appendix B, problem 1), moved to problems that had the potential to involve ordered outcomes (e.g., Appendix B, problem 5), and finally involved problems that could involve binomial multiplication (e.g., Appendix B, problem 3). When appropriate, students were given concrete materials (e.g., 13 cards) and asked to perform experiments (e.g., draw a card, replace it, and draw a card again). Problem contexts were tailored to students' experiences so that, for example, if they were asked to solve a problem involving possible meals that a restaurant could serve (Appendix C, problem 1), the interviewer would ask them what their favorite restaurant was (or if they did not go to restaurants what their parents cooked at home). This ensured a level of familiarity for the students with the contexts about which they were being asked. On occasion, the numbers in a

[2] Given school schedules, five students participated in three forty-minute interviews instead of two hour-long interviews.

problem were varied for a student; the numbers that appear in Appendix B and C represent those most commonly used.

For the first problem, students were asked to represent their thinking in some way other than simply stating an answer. Depending on their responses, the interviewer introduced a list, a tree diagram, and/or an array as possible ways to represent their solution to a problem. The interviewer did so through a series of prompts where, for example, a student who represented the first problem pictorially (e.g., by drawing pants, shirts, and each outfit) was asked a series of questions like: I: "Which outfit is this one?"; S: "The one with the red shirt and blue pants."; I: "Could you use two letters to represent that outfit?"; S [Writes "RB"]; I: "Could you write each outfit using two letters?"; S [Creates a list]. As part of structuring the representational process, the interviewer organized questions that helped students first create a list, then a tree diagram, and subsequently an array although the interviewer adjusted this sequence to students, depending on what seemed sensible to them and what representations they choose to use.

To introduce representing problems as arrays, the interviewer asked students the following question: "Can you make an array like the Cartesian coordinate system?" The interviewer used this prompt because the Cartesian coordinate system was a representation that students used in their regular classroom experiences. All of the students were able to use this question to produce an array for at least one Cartesian product problem. As the data will illustrate, this did not guarantee that students understood the arrays that they produced in conventional ways. For example, not all students understood that a point in an array represented a pair even though they could produce what looked like a conventional array to represent the set of outcomes. The interviewer was attentive to this issue and designed questions to investigate how students understood the representations they produced.

4.3 Data Analysis

The research team engaged in on-going analysis of the interviews at the time of data collection. On-going analysis included meeting weekly to discuss the notes that each research team member had taken during interviews and to watch segments of video to establish working models of the students' reasoning (Steffe & Thompson, 2000). After the data collection phase was over, the research team engaged in retrospective analysis. As part of this process, the mathematics education researcher transcribed all interviews, wrote low-inference data summaries for each problem that a student solved, and wrote a memo to document conjectures about the data (Corbin & Strauss, 2008; Saldaña, 2013). The transcripts, data summaries, and memos were then discussed among members of the research team in order to triangulate interpretations of the data (Mathison, 1988). The mathematics education researcher cycled through the data looking first at each individual student's solutions across all problems, then comparing all students who were operating with similar multiplicative concepts, and finally comparing students operating with different multiplicative concepts. The aim of this work was to provide a consistent and coherent account of the students' reasoning within individual participants and across all participants (Steffe & Thompson).

5. Data Analysis

In the Data Analysis, I investigate each of the three different reasoners starting with MC1 students, progressing to emergent MC2 students, and then elaborated MC2 students. For each reasoner, I first examine how they solved Cartesian product problems, where I focus on the extent to which they carried out pairing operations in their solution of problems. I then examine how they created arrays, where I focus on the multiplicative relationships they established among the units on the axes of the array and the pairs in the interior of their arrays.

5.1. Data on MC1 Students

5.1.1. MC1 students' solution of Cartesian product problems

MC1 students did not always use a pairing operation in their solution of Cartesian product problems. To illustrate this issue, I compare and contrast two data excerpts, one from Darryl and one from Alana. Darryl's data excerpt illustrates a student who *did not* use a pairing operation, while Alana's solution illustrates a student who *did*. Darryl's data excerpt comes from his solution of the Card Problem.

> *The Card Problem*: You have the ace through king of hearts (13 cards). Your friend has the ace through king of clubs (13 cards). Use an array to show all of the possible 2-card hands you could make that consist of one heart and one club. On your array show the number of 2-card hands that have two face cards (Jack, Queen, King), that have exactly one face card, and that have no face cards. Use the sections of your array to determine the total number of 2-card hands you can make.

To solve the Card Problem Darryl and the interviewer made two two-card hands using playing cards. Then Darryl predicted he could solve the problem by multiplying "thirteen times thirteen", and made the beginning of an array (Figure 3). The interviewer asked Darryl if he could identify where in his array there were two-card hands that had two face cards. Darryl identified that the numbers 11, 12, and 13 on each axis in Figure 3 represented face cards, but he could not identify which points in his array represented two-card hands with two face cards. The interviewer asked Darryl to get the three spade face cards from his deck of playing cards, and the interviewer got out the three heart face cards from his deck of playing cards. The intent was to have Darryl use the cards to think about how many two-card hands had two face cards. The following discussion ensued.

Figure 3. Darryl's array[3]

Data Excerpt 1: Darryl Makes a Units Coordination to Solve the Card Problem

I: Okay, so I got to get these cards out [interviewer gets out his three face cards]. Right so those would be the three (cards) in mine (my hand). So can you tell me--

D [interrupting the interviewer]: This [takes the jack of spades] could match up with those three cards [indicating the jack, queen, and king of hearts], this [takes the queen of spades] could match up with those three cards [indicating the jack, queen, and king of hearts], this [takes the king of spades] could match up with those three cards [indicating the jack, queen, and king of hearts]. So it would be three, six, nine.

I: Yeah, that is exactly right. Oh by the way, how many face cards would those hands have? Do they have two? Do they have one face card?

D: One.

I [surprised]: They have one?

[3] This is a recreation of Darryl's array to reflect what it looked like at the point in the interview when the conversation took place.

D: No. They have three each.

I infer Darryl assimilated the situation using two composite units—three and three. I make this inference because he operated with his three cards and the interviewer's three cards to produce the correct numeric response of nine. However, to the interviewer's surprise, he responded that there were "one" and then "three" face cards in each "two-card hand." Because of Darryl's response to this question, I infer that he did not use a pairing operation to solve the problem. Instead I infer he engaged in a units coordination, inserting three units of the first composite unit into each of the three units of the second composite unit to create a unit of three units in activity each time he took one of his spades and envisioned "matching it up" with the interviewer's three hearts (Figure 4). I make this interpretation because he seemed to consider the interviewer's question as about the containing unit (*one* spade for every three hearts), and then about the number of units that it contained (one spade for every *three* hearts). This indicated that up to this point in his solution he had not established two-card hands using a pairing operation, even though he did interpret the situation as involving multiplication. This way of operating produced a unit structure similar to ones he created in isomorphism of measures problems like the Donut Problem: three of his cards for every one of the teacher's cards, which is similar to three donuts for every one package. Thus, in his solution of this problem, he did not produce a more complex unit structure that can be produced in product of measures problems, and his meaning for multiplication in this situation did not entail a pairing operation.

Figure 4. Creating a unit of three units in activity

His way of operating to solve the problem was revealing because he had produced pairs in his solution of *earlier* problems, but he had not produced pairs in his solution of this problem up to this point. Thus, his response provides indication that without producing pairs as part of his activity he could not take them as a given in a situation. This interpretation illuminates why Darryl did not initially locate in his array the two card hands that had two face cards—he could re-create how an array looked, but, at this time in his solution of the problem, the points in his array did not actually represent pairs because he had not created pairs as part of his activity. This meant he could not "see" features of pairs like how many face cards were in a two-card hand.

Alana's solution of the Dice Problem contrasts with Darryl's solution of the Card Problem in that she used a pairing operation as part of her solution.

> *The Dice Problem:* You have a small die. I have a large die. We each roll our die. An outcome is the number on your die paired with the number on my die. How many possible outcomes could we get?

To solve this problem, Alana created a tree diagram (Figure 5) while sub-vocally listing number pairs ("one-one, one-two," etc.). Then the following discussion took place.

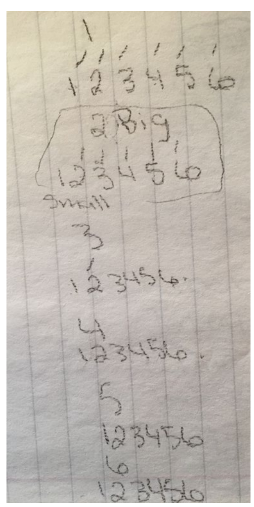

Figure 5. Alana's tree diagram for the Dice Problem[4]

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[4] This is an atypical tree diagram in that the "trunk" is often to the left and the "leaves" to the right whereas Alana created the "trunk" above and the "leaves" below. She also did not make all of the connecting lines and did not make the connecting lines meet in a single point. Another way to interpret her representation is as an abbreviated list

Data Excerpt 2: Alana's solution of the Dice Problem

I: Good. So do you think that takes care of everything?

A: Not yet. [Alana has created all outcomes where the number from the big die is in the first position and the number from the small die is in the second position. She indicates that she is uncertain whether it would be considered a different outcome if the number from the small die were in the first position of the outcome and the number from the big die in the second position. Alana and the interviewer discuss this issue for 3 minutes and 37 seconds. Alana decides not to count these as different outcomes.]

I [After their discussion, the interviewer asks Alana about Figure 5]: Will you tell me a little bit about just these ones [points to Figure 5]? In other words, can you tell me how many total you ended up with? Do you know how many total you ended up with?

A [begins to count the numbers below the number one, stopping when she gets to the number three that is under the number one. She starts her count over, sweeping her pencil between the number one and each number below. She continues this action until she finishes by sweeping her pencil from the number six on top to the number six below.]: Thirty-six.

I: Awesome. How did you do that?

A: I matched the numbers up…at first I was just counting straight. I realized I had to do it like this [points to a previous problem]. I just matched the numbers up like one-one, one-two, one-three, one-four, one-five, and one-six.

Based on her tree diagram, I infer that Alana, like Darryl, assimilated the situation using two composite units. However, this excerpt contrasts with Darryl's in that Alana operated differently

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where, for example, she stopped writing "1" above each line as she made the pairs, and subsequently stopped writing all (and eventually any) of the connecting lines between two numbers.

on the composite units—she paired the first unit of one composite unit with each unit of the second composite unit ("I just matched the numbers up like one-one, one-two, one-three…") (Figure 6). Moreover, it provides further evidence about the status of pairs for MC1 students.

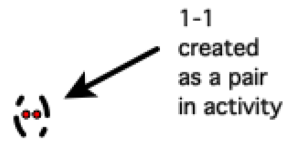

Figure 6. Alana creates a pair in activity.

Alana created the pairs, 1-1, 1-2, 1-3, etc. when she first created Figure 5, as evidenced by her subvocally saying each pair. She and the interviewer then had a conversation for 3.5 minutes about ordered outcomes. When Alana returned to Figure 5, she initially began counting the numbers underneath the number one ("at first I was just counting straight"), but she stopped this count, and then swept her pencil from the top number one to the bottom number one, the top number one to the bottom number two, etc. until she finished her count and said, "thirty-six." This sequence of events provides indication that Alana created pairs *in activity*: She created the pairs when she first created Figure 5, but a few minutes later, after the discussion with the interviewer, she had to create the pairs again in order to count them.

These data excerpts illustrate that MC1 students seemed to be constrained to creating pairs as part of their activity. In Darryl's case, without creating pairs as part of his activity, the points in his array did not stand in for two-card hands as was evidenced by his not being able to locate where the two card hands were that contained two face cards. In Alana's case, to count the pairs she had to create them again. Darryl's solution does, however, indicate that MC1 students could interpret the situations as involving multiplication without creating pairs; they simply interpreted the problems in a way that was similar to an isomorphism of measures problem, and they produced a unit structure similar to the one that they would produce in such a problem.

5.1.2. The multiplicative relationships MC1 students' established with arrays

The first two data excerpts illustrated that MC1 students created pairs in activity in their solution of Cartesian product problems. Central issues, then, were how students used this operation as they created arrays, what additional operations students might use when they produced pairs in the context of creating arrays, and what *multiplicative relationship* did this imply the students created. Carlos's solution of the Restaurant Problem highlights these issues.

> *Restaurant Problem*: Steak and Shake has 12 different main dishes and 6 kinds of shakes. A meal is one main dish and one kind of shake. How many different meals does Steak and Shake offer?

Figure 7 shows Carlos's completed array, but the data excerpt is about its creation.

Data Excerpt 3: Carlos solves the Restaurant Problem

C: Could we name the shakes like A, B, C?

I: Yep. You can do that certainly.

C [puts the letters A through N on the vertical axis for the main dishes. He counts the number of letters he has written.] So there are twelve main dishes, and six shakes?

I: Mm-hmm.

C [Erases the letters M and N, leaving the letters A through L on the vertical axis. He puts the letters A through F on the horizontal axis.]: So shake A [points to the A on the vertical axis]. No main dish A [keeps his finger on the A on the vertical axis] with shake A. So I'd put a point right there [puts a point in the lower left corner of his array]. So this one [points to B on the vertical axis] with this one [points to the A on the horizontal axis, and puts a point directly above the point in the lower left corner]. This one with this one [Carlos puts a third point directly above the first two]. It would keep going. [Carlos continues pointing to each main dish and shake A as he makes the points in his array that represent meals with shake A.]

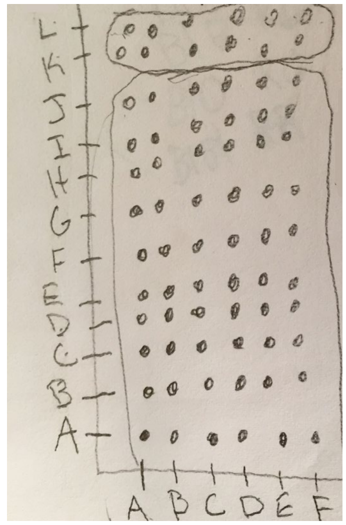

Figure 7. Carlos's array for the Restaurant Problem

Carlos, like Darryl and Alana, provided indication that he assimilated the situation using two composite units—he represented twelve along one axis of his array and six along the other axis of his array. To create the points in his array, Carlos located the appropriate letter on each axes with his fingers, moved his fingers to where they intersected, and created a point. I take these physical actions as evidence that he established a pair independently from, but related to each unit of one that constituted them, where the units of one were represented along the axes. I infer that this way of operating entailed using his pairing *and* disembedding operation in activity, where a disembedding operation is the operation that enabled him to establish a pair as independent from, but related to the units of one that constituted it (Figure 8). Throughout the interviews, Carlos consistently used both a pairing and disembedding operation in his solution of Cartesian product problems. However, not all of the MC1 students used both operations. For example, Alana almost exclusively relied on lists to record pairs (e.g., “1A”, “1B”, etc.), but did not make a clear record of the units of one as independent from the pairs. This provided counter-indication of her use of a disembedding operation in her solution of problems even though she did consistently use a pairing operation (compare Figure 6 and Figure 8).

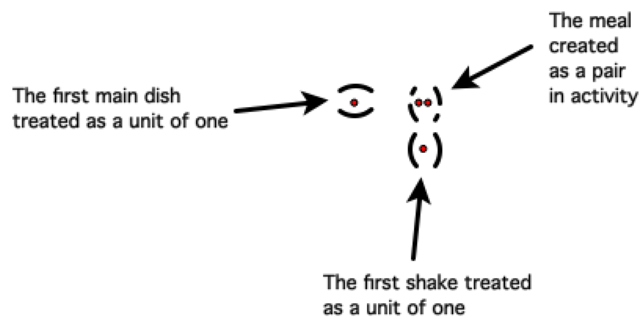

Figure 8. Creating a multiplicative relationship among a unit of one, a unit of one, and a pair in activity

When students used both a pairing and disembedding operation, I considered these ways of operating evidence that they could establish a multiplicative relationship between a unit of one, a unit of one, and a pair in activity (Figure 8). I consider this relationship *multiplicative* because there was evidence that a student maintained the units of one independently from the pair. Thus, this way of operating could be the basis for establishing the multiplicative identity that one times one is equal to one. This inference does *not* mean that MC1 students necessarily experienced the situation as involving multiplication (e.g., “It’s twelve times six”); in fact they were more likely to experience the situation as involving multiplication when they engaged in a units coordination, as Darryl did, than when they used a disembedding and pairing operation. They were more likely to do so because the use of a disembedding and pairing operation produced a more complex unit structure, and their “world” seemed to be consumed by the creation of the multiplicative relationship in Figure 8.

Evidence that MC1 students did not establish a more complex multiplicative relationship was that they did not initially anticipate how many pairs they could create in a particular row or column of their arrays. This issue can be illustrated in the continuation of Carlos's solution of the Restaurant Problem. After he finished making the points that represented a meal that contained shake A, the following interaction occurred.

Continuation of Data Excerpt 3: Carlos solves the Restaurant Problem

C [finishes making the first column of points in Figure 7. Looks at the interviewer]: So, now (main dish) A with (shake) B.

I: Sounds good. Before you do that, how many did you make [points to the column of points in Figure 7 that represent all meals with shake A]? Do you know?

C [begins to count the points, but does so haphazardly. Starts over and counts more precisely]: One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve.

The fact that Carlos counted the number of points in his array provides indication that he did not know that there were twelve meals represented in the first column of his array. As the interviews progressed, Carlos did make the association that the number of pairs he created would be equal to the number of units in one of the composite units (e.g., 12 main dishes would create 12 meals) and he could use this association to anticipate the number of pairs that he would make with a particular item (e.g., the first shake). However, this understanding was not immediate for any of the MC1 students, indicating that it was unlikely that they created this as a result of creating a more complex multiplicative relationship. Instead it was simply an association they made as a result of experience with the problems.

5.2. Emergent MC2 Students

5.2.1. Emergent MC2 students' solution of combinatorics problems

When emergent MC2 students solved combinatorics problems, there was strong evidence that they had interiorized pairs. This evidence included that they could operate as if pairs were part of a situation prior to creating *all* of them. Kai's solution to the Vending Machine Problem demonstrates this issue.

> *Vending Machine Problem:* A vending machine has 6 kinds of chips and 4 kinds of candy bars. A snack is 1 kind of chip and 1 kind of candy bar. How many possible snacks could you get from the vending machine?

Data Excerpt 4: Kai's solution to the Vending Machine Problem

K [The interviewer has asked Kai to name six different kinds of chips and four kinds of candy bars. As she names the six kinds of chips, she puts up the five fingers on her right hand and her left thumb. As she names the four candy bars, she puts up the four remaining fingers on her left hand.]: So you could go hot lays [puts her two pinkies together—her right pinky represents the hot lays chips and her left pinky the first candy bar] with all of those [sweeps her right pinky across the four fingers on her left hand that represent candy bars] so that is one (snack) [returns her right pinky to rest on her left pinky], four (snacks). You could go one (chips) with four of them (candy bars) [touches her right thumb, which represents the second chips, to her left pinky, which represents the first candy bar]. Then you take this one (chips) with four of them (candy bars) [touches her right index finger, which represents the third chips, to her left pinky, which represents the first candy bar]. Take this one (chips) with four of them (candy bars) [touches her right middle finger, which represents the fourth chips, to her left pinky, which represents the first candy bar]. You take this one (chips) with four of them (candy bars) [touches her right ring finger, which represents the fifth chips, to her left pinky, which represents the first candy bar]. So that's [stares into space] ... twenty-four [concludes this without touching her left thumb, which represents the last bag of chips, to her left pinky, which represents the first candy bar].

I: Yeah that is great. So tell me what you did. That was impressive.

K: Okay. Since I got four candy bars, and I got six bags of chips. So, and like just, they don't just go with like one [touches the pinky on her right hand to her pinky on her left hand] like you could go with hot lays with any one of them [runs the pinky, which represents hot lays chips, on her right hand across the four fingers of her left hand]. You could just take the one (chips) by the four (candy bars), and then you would just do that six times, and six times four is twenty-four.

I infer that Kai assimilated the situation using two composite units based on the fact that she put up six fingers and four fingers. She then operated on the composite units; she put her two pinkies together as indication that she created one pair with the hot lays chip and the first candy bar. She was then able to take creating this pair ("one (snack)") as indication that she could create four pairs with the hot lays chips ("four (snacks)"), but she did not need to actually make these pairs as part of her activity (Figure 9). An alternate interpretation would be that she simply made a units coordination to make a unit of four units (like Darryl did in Data Excerpt 1) rather than envisioned making four pairs. I don't make this interpretation because throughout her solution of the problem she referred to the results as pairs (e.g., "It's the hot lays with snickers").

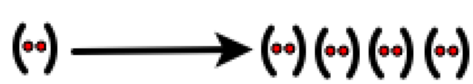

Figure 9. Creating one pair (on the left) implies that she could create four pairs (right)

She continued her solution by repeating her actions with five of the six kinds of chips, putting her right thumb with her left pinkie, then her right index finger with her left pinkie, etc.

She then stopped and looked off into space without creating any pairs for the sixth bag of chips, which I interpret as indicating that she knew she would make the same number of pairs with the final bag of chips without actually having to make any of these pairs. My inference is that she could operate in this way because she had interiorized pairs, which enabled her to operate as if the pairs were part of the situation without actually having to create them as part of her activity; in putting together her two pinkies she established a single pair and then without needing to establish the rest of the pairs with the "first bag of chips" she could operate as if those pairs were part of the situation. This differed from MC1 students in that MC1 students were not able to reason about the pairs when they had not made them in immediate past activity (e.g., Darryl's solution of the Card Problem or Alana's solution of the Dice Problem). In contrast, MC2 students could operate as if the pairs were part of the situation without making them in activity.

5.2.2. The multiplicative relationships emergent MC2 students' established with arrays

In the context of developing arrays, there was evidence that emergent MC2 students had interiorized a multiplicative relationship among a unit of one, a unit of one, and a pair (see Figure 8). This meant that they could take as a given that individual points in their arrays represented pairs and fluidly relate the pairs in their array to the units that constituted them, which were represented along the axes of the array. Based on my analysis of MC1 students, I infer that the interiorization of this relationship was a result of using both a pairing and disembedding operation. Jada's solution of the Restaurant Problem illustrates how a student operated when they had interiorized this multiplicative relationship.

Restaurant Problem: A restaurant serves 14 kinds of steaks and 6 kinds of salad. A meal is one steak and one salad. How many different meals are possible to order?

Data Excerpt 5: Jada's array for the Restaurant Problem

I [To create Figure 10, Jada writes the numbers along each axis, and fills in the points]: So what does that dot represent [points to the sixth salad paired with the first steak]?

J: One of the salads with one of the steaks (entrées).

I: Which salad?

J: Number six

I: Which entrée (steak)?

J: Number one.

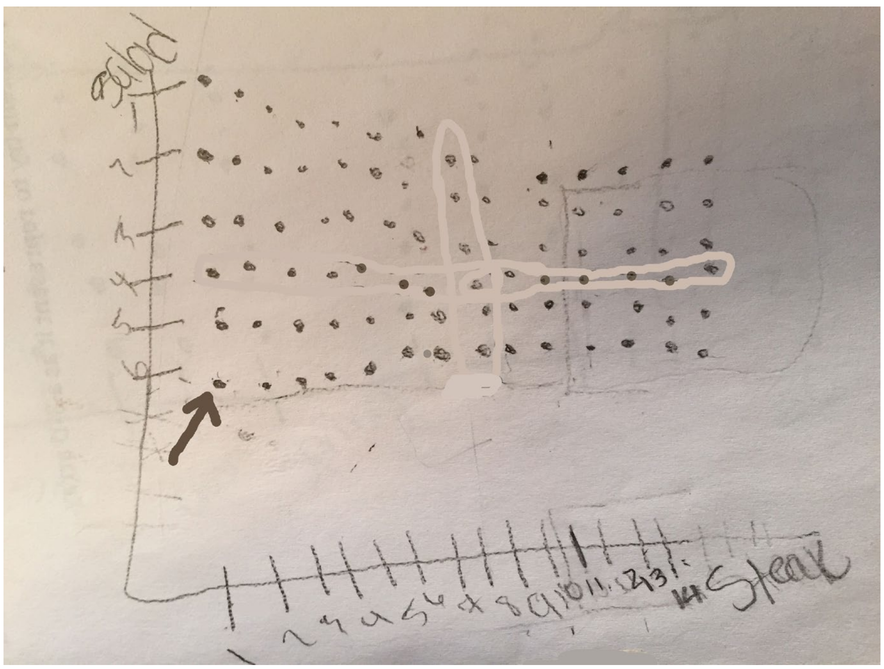

Figure 10. Jada's array for the Restaurant Problem

To create her array, Jada did not engage in verbal listing of pairs or any physical action relating the units on the axes to the points in the interior (e.g., finger motions). After creating her array, she immediately identified a point in her array as referring to "one of the salads with one of the steaks" even though she had not just created this particular salad and steak pairing in immediate past activity. Moreover, her language was general in that she did not identify which salad and which steak the point represented (although as the follow-up questions indicated, she could do this fluidly). I interpret her language as indicating that she assumed that this point, like all of the other points in her array, would represent one salad paired with one steak and that this pair was connected to units on the axes, which could help her determine which salad and which steak. My inference is that this way of operating indicated the interiorization of a multiplicative relationship among a unit of one, a unit of one, and a pair—namely emergent MC2 students did not necessarily need to use a disembedding and pairing operation as part of their activity, but rather could see the situations as involving this relationship. The main differences between MC1 and emergent MC2 students that indicated that emergent MC2 students had interiorized this multiplicative relationship were: a. the generality of the language emergent MC2 students used; b. the fluidity with which emergent MC2 students identified points as pairs and related them to the units on the axes; and c. the lack of indication that they were operating with the units on the axes to re-create pairs once an array had already been created.

A question for emergent MC2 students was whether they established a multiplicative relationship among a unit of one, a unit of units, and a unit of pairs in activity (Figure 11). Doing

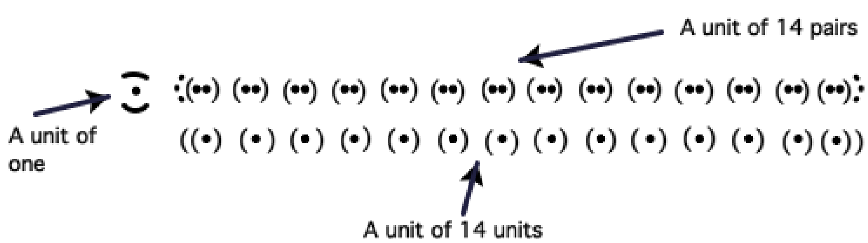

Figure 11. A multiplicative relationship among a unit of one, a unit of 14 units, and a unit of 14 pairs

so would mean that they established a row and/or column in an array as a multiplicative relationship. For example, they considered that one salad times fourteen entrees was equal to fourteen meals, and could reason about the pairs contained in the row and/or column when working with more than a single pair. From the perspective of operations, this would mean that they disembedded the first unit from one composite unit they used in assimilation (e.g., 14), disembedded the first unit from the other composite unit they used in assimilation (e.g., 6), paired these two units together to create a single pair, which they could take as implying all fourteen pairs they could create, and then engaged in a units coordination, inserting these fourteen pairs into a containing unit to create a unit of 14 pairs in activity. From the perspective of operations, then, the key question was whether they engaged in a units coordination where they inserted the 14 pairs into a containing unit.

To investigate whether emergent MC2 students established this multiplicative relationship, I present two data excerpts. The first is a continuation of Jada's solution of the Restaurant Problem, which illustrates that emergent MC2 students could anticipate the number of pairs that would be in a row and/or column. The second illustrates that despite being able to anticipate the number of pairs in a row and/or column there was counter-indication that emergent MC2 students established a row or column as a multiplicative relationship among a unit of one, a unit of units, and a unit of pairs where they could reason about the pairs.

Continuation of Data Excerpt 5: Jada's array for the Restaurant Problem

I [Jada is looking at Figure 10]: So how many different salads could the first entree go with?

J: Six.

I: What about the second one?

J: Six.

I: How many different entrees could go with the sixth salad?

J: Fourteen for each different salad.

I: For each of them so that would be the same as salad one?

J: Yeah. Fourteen different steaks (entrées) could go with salad number one, two, three, four, five, and six.

….

I: Where are all of the entrees that could be paired with the fourth salad? [J circles the row in her array (shown in Figure 12)]. What about all the salads that could be paired with the eighth entree? [J circles the column in her array (shown in Figure 12)].

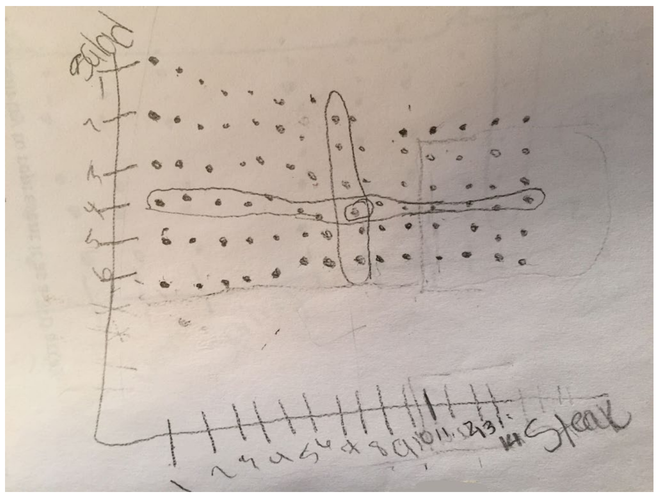

Figure 12. Jada's array for the Restaurant Problem

Because emergent MC2 students were able to operate as if pairs were part of a situation even when they carried out a minimal amount of pairing activity (as was illustrated in Kai's solution of the Vending Machine Problem), they were able to identify the number and location of pairs in a row or column of their arrays. Jada knew the total number of meals that were with the first entrée (six); she also knew the total number of meals that were with the sixth salad (fourteen); and she could easily locate all meals with a particular entrée or salad in her array. This contrasted with MC1 students, as was illustrated in the continuation of Carlos's solution to the Restaurant Problem where he was unsure about how many meals were in the first column of his array after filling in the points for those meals.

Although emergent MC2 students were able to locate rows and/or columns in their arrays and state the number of pairs in a given row and/or column, there was counter-indication that they established a row and/or column as a multiplicative relationship among a unit of one, a unit of units, and a unit of pairs in activity (Figure 11). Jada's solution of the Card Problem illustrates this issue.

> *Card Problem.* You have the ace through seven of hearts. I have the ace through seven of spades. We make a two card hand by drawing one card from your hand and one card from my hand. How many possible two card hands could we make?

Jada solved the Card Problem (with 7 hearts and 7 spades) and made the array shown in Figure 13a. The interviewer then asked her to predict the number of new two card hands she could make if each person added one card to their hand. Jada predicted that it would be "seven plus one" new two-card hands. The interviewer then asked Jada to fill in the new points on her array, which she did, commenting that the point that represented the "plus one" was the eight of spades and the eight of hearts (Figure 13b). Jada did not initiate any further activity towards solving the problem with indirect questioning, so the interviewer suggested directly for her to consider other two-card hands she could make with the eight of hearts. After this direct suggestion, Jada created the other two card hands she could make with the eight of hearts, and filled in the points for those two card hands on her array. When she filled in the points, she included a second point in her array for the eight of hearts and the eight of spades (Figure 13c). She erased this point after a direct question from the interviewer about it. The following exchange then took place.

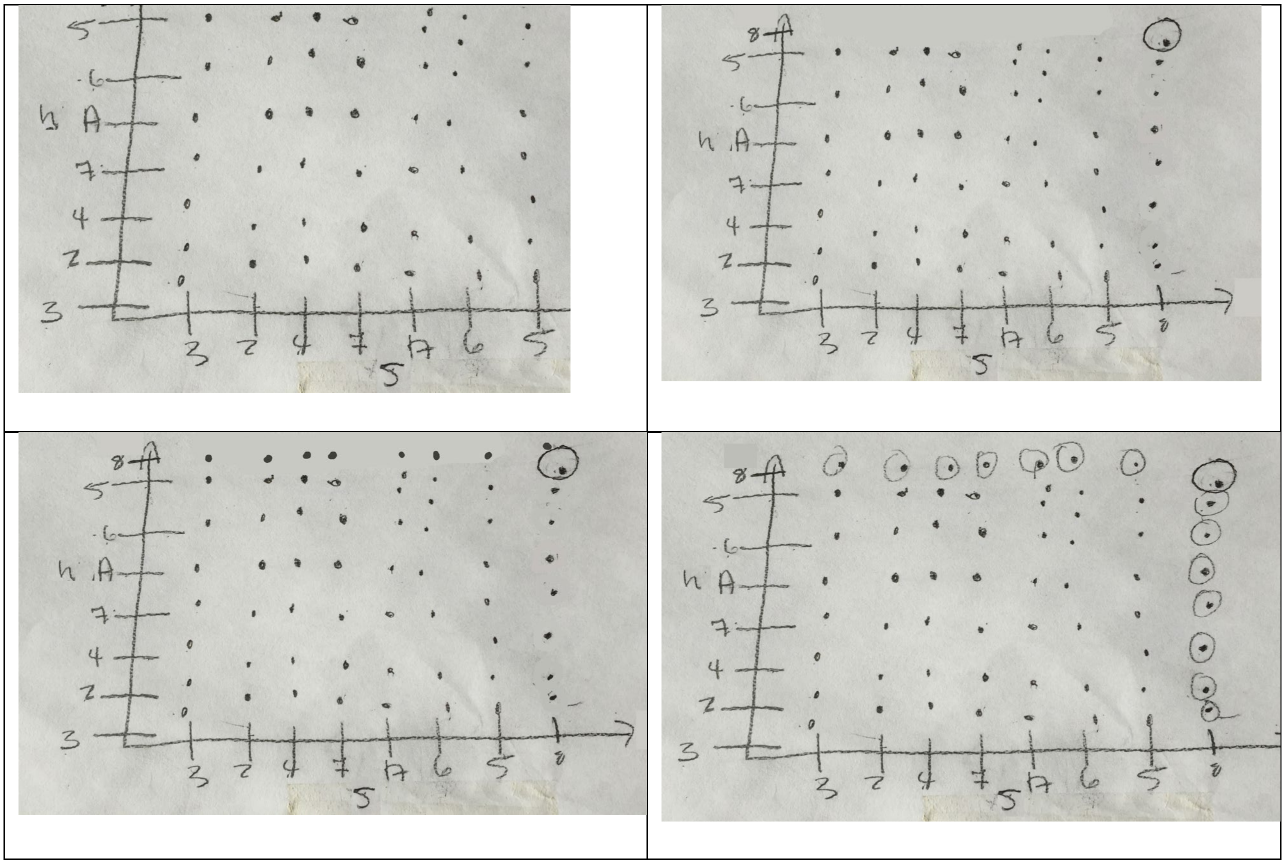

Figure 13a (top left), 13b(top right),13c(bottom left) & 13d (bottom right): Jada's array for the Card Problem

Data Excerpt 6: Jada continues her solution of the Card Problem

I: So how many new two-card hands? Circle all the new two-card hands that you got. [Jada circles the individual two card hands (Figure 13d)]. Tell me how many you got before you count.

J: That would be sixteen.

I: You're close. How did you get sixteen?

J: Eight plus eight.

I: Okay so you think it might be eight and eight. Will you now count them and check?

J [pauses, looks at the interviewer, and says to emphasize her prediction]: Hold on, before I count them, I think it's sixteen.

I: You think it's sixteen.

J: One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen. [Jada recounts to make sure]. Yeah, fifteen.

I: So tell me why it's fifteen and not sixteen.

J: Cause that extra one [points to the corner dot] was counted as the rest of the row [points up and down vertically].

There were two indicators that Jada did not create a row and/or column in her array as a multiplicative relationship among a unit of one, a unit of 8 units, and a unit of 8 pairs in activity. The first indicator was that she initially created a second point in her array for the eight of hearts and the eight of spades. This provides indication that she did not maintain the eight pairs she had created with the eight of spades as part of the situation once she shifted her attention to the pairs she could create with the eight of hearts. Thus, she did not establish the column as a multiplicative relationship among a unit of one, a unit of 8 units, and a unit of 8 pairs where the pairs could be reasoned about in the continuation of her solution to the problem. The second indicator was that she predicted that there would be a total of 16 new two-card hands even after she established that the eight of hearts and eight of spades should not be represented twice. She remained relatively certain of her prediction even after the interviewer suggested that 16 was "close"; she paused, looked directly at the interviewer, and re-asserted her prediction. She then counted the points twice to verify that her prediction was off by one. This suggested that she did not establish a row or column as a multiplicative relationship among a unit of one, a unit of units, and a unit of pairs that she could use in her reasoning to adjust the prediction she made about the total number of new pairs she would create.

These two data excerpts illustrate that emergent MC2 students could readily locate rows or columns in an array, but that it was difficult for them to reason about the pairs contained in the rows or columns when this reasoning involved establishing multiple pairs in the process of solving a problem. In Jada's case the issue was "seeing" that the eight pairs she would create with the eight of hearts and the eight pairs she would create with the eight of spades both contained the eight of hearts and eight of spades pair.

5.3. Elaborated MC2 Students

5.3.1. Elaborated MC2 students' solution of combinatorics problems

Like emergent MC2 students, elaborated MC2 students could engage in a relatively small number of pairing operations prior to concluding the total number of pairs they could create. Unlike emergent MC2 students, elaborated MC2 students used different language to describe their solution of problems. Urbano, for example, routinely made statements like, "one times seven equals seven, right? Yeah", when solving problems. This language explicitly made reference to a multiplicative relationship among one unit, seven units, and seven pairs. This language can be compared with emergent MC2 student's like Kai's language in Data Excerpt 4 where she referenced the pairs she produced ("one (snack)" and "four (snacks)") or referenced the units she would use to produce the pairs ("one (chip) by four (candy bars)"), but she did not reference the two together in an explicit statement of multiplication. This difference in language in their solution of Cartesian product problems was one initial indicator that *elaborated MC2 students* established a row or column in an array as a multiplicative relationship between a unit of one, a unit of units, and a unit of pairs in a way that enabled them to reason about the pairs in a row and column when they were asked to reason about more than one pair.

5.3.2. The multiplicative relationships elaborated MC2 students' established with arrays

Tiana's solution of the Sub Sandwich Problem illustrates that elaborated MC2 students established a row or column in an array as a multiplicative relationship among a unit of one, a unit of units, and a unit of pairs in activity.

> *Sub Sandwich Problem:* Subway has 6 kinds of bread and 7 different kinds of meat. You can make a sandwich by choosing 1 kind of meat and 1 kind of bread. Draw a picture to help you determine how many total sandwiches you can order.

To respond to this problem Tiana immediately stated that it would be a total of 42 sandwiches because she could make 7 sandwiches with each of 6 breads. The interviewer asked her to find a way to "show" the sandwiches to which she asked if she could use letters to represent breads and numbers to represent meats, and then created an array (Figure 14). Once Tiana created her array the interviewer asked her the following extension of the Sub Sandwich Problem.

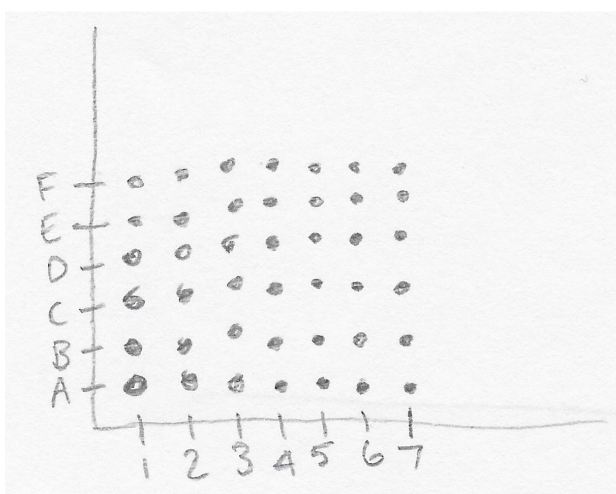

Figure 14. Tiana's array[5]

[5] This is a re-creation of Tiana's array based on what it looked like at the time she created it.

Extension of the Sub Sandwich Problem: Suppose Subway added 1 extra bread and 1 extra meat to their menu (7 breads total and 8 meats total). How many *new* (more) sandwiches could you make?

Tiana suggested G represent the seventh bread and the number 8 represent the eighth meat at which point the interviewer covered her array and they had the following interaction.

Data Excerpt 7: Tiana's solution of the Extension of the Sub Sandwich Problem

I: So how many new ones would there be?

T: It'd be a row of eight. That is probably it.

I: You want to say out loud what they'd be?

T: G-1, G-4 like that.

I: Would you say all of them?

T [interprets the interviewer's request as saying *all* of the sandwiches not just all of the *new* sandwiches]: A-1, A-2, A-3, A-4, A-5, A-6, A-7, A-8; B-1, B-2, B-3, B-4, B-5, B-6, B-7, B-8; C-1, C-2, C-2, C-4, C-5, C-6, C-7, C-8; D-1, D-2, D-3, D-4, D-5, D-6, D-7, D-8; E-1, E-2, E-3, E-4, E-5, E-6, E-7, E-8; F-1, F-2, F-3, F-4, F-5, F-6, F-7, F-8; G-1, G-2, G-3, G-4, G-5, G-6, G-7, G-8.

I: And how many of those are new? In other words are not ones that you already had?

T: Eight really. You mean you want me to multiply it across (to find the total number of sandwiches)?

I: I wonder if you can list them out (verbally). What the new ones were?

T: Wait. I'm thinking it's fourteen more. G was my new bread and it went with meats one through eight for eight (sandwiches). Then I had A-8 [puts up her right thumb], B-8 [puts up her right index finger], C-8 [puts up her right middle finger], D-8 [puts up her right ring finger], E-8 [puts up her right pinky], F-8 [puts up her left thumb], which is six more.

I [her array is still covered]: Whoa. Cool. Where would they be on your array?

T: They'd be like this. [Tiana traces her finger horizontally showing a row and vertically showing a column. Then at the request of the interviewer fills the points in on her array.]

The interviewer was specifically interested in investigating whether Tiana could solve this problem in the absence of seeing her array. Tiana initially predicted there would be eight new sandwiches. The interviewer pressed Tiana to list what the eight new sandwiches would be, and Tiana listed two of these new sandwiches (G-1 and G-4), providing indication that the eight new sandwiches she considered were all the ones with the new bread. The interviewer pressed again for her to list "all of" the sandwiches, intending to have her list *all of the new* sandwiches. Tiana, however, interpreted this request as one to list all of the sandwiches that she could make. After she verbally listed all of the sandwiches, the interviewer returned to the question of how many of these were new. Tiana again said there would be eight, but then seemed to have an insight saying, "Wait. I'm thinking it is fourteen more.", explaining that there could be eight sandwiches with bread G, and then listing the six sandwiches with meat "8", excluding sandwich "G-8." Unfortunately, the interviewer did not directly ask Tiana about why she only counted the sandwich "G-8" once. However, Tiana did not even seem to consider counting it twice—she said there would be 14 new sandwiches. I take her statement as indication that prior to listing the sandwiches she could make with meat "8", she already knew there would be six new sandwiches beyond the eight she had already determined would be new. Her verbal listing confirmed that she intended to exclude sandwich "G-8" because she apparently already knew that she had counted it as one of the 8 sandwiches with bread "G." She then traced with her finger where the new sandwiches would be, indicating that she knew they would form a row going across and a column going down, and filled in her array with the appropriate points—making only one point for the sandwich “G-8.”

Like emergent MC2 students (e.g., Jada), elaborated MC2 students like Tiana initially predicted that there would only be new sandwiches with one of the two new items introduced (i.e., either the new bread or the new meat, but not both). Unlike emergent MC2 students, elaborated MC2 students resolved this question without counting twice the pair that contained both of the new units (both the new bread and the new meat). Moreover, elaborated MC2 students were able to resolve this issue without the support of seeing their arrays: Tiana was able to envision where the points of the array would be prior to filling them in on the array. This quality indicated that Tiana, and other elaborated MC2 students, were able to coordinate the pairs in a row and column simultaneously even when they were reasoning with more than a single pair in a row and column. I explain this difference as follows: Tiana, and other elaborated MC2 students, engaged in a units coordination where they inserted the eight pairs into a containing unit which established a row/column as a multiplicative relationship among a unit of one, a unit of units, and a unit of pairs in activity (see Figure 11 above). I considered them to establish this relationship *in activity* because Tiana, and other elaborated MC2 students, made the relationship as part of the activity they produced in the situation. In the case of Tiana, she verbally listed all of the sandwiches prior to identifying that there would be 14 new sandwiches—her activity of listing all of the sandwiches seemed an essential part of her identifying what all of the new sandwiches would be.

6. Discussion

6.1. Discussion of Findings

The differences among students from this study are summarized in Table 2.

Table 2. Summary of findings

	Use of pairing operation in Cartesian product problems	**Multiplicative relationships established**	**Implications for arrays**
MC1 students	Created pairs in activity, which meant that to consider features of pairs, they had to actually make them	Could establish a multiplicative relationship among a unit of one, a unit of one, and a pair in activity using their disembedding operation in activity	Took points in arrays as pairs that were related to the units on the axes in activity, which meant that points in arrays as pairs had a transitory status
Emergent MC2 students	Interiorized pairs, which meant that they engaged in significantly fewer pairing operations than MC1 students in the solution of Cartesian product problems	Interiorized a multiplicative relationship among a unit of one, a unit of one, and a pair. Did *not* create a multiplicative relationship between a unit of one, a unit of units, and a unit of pairs in activity	Took points in arrays as pairs and were able to quickly identify the number of pairs in a row and/or column. Made conflations when reasoning simultaneously with rows and columns when they were reasoning with multiple pairs.
Elaborated MC2 students	Similar to emergent MC2 students	Engaged in a units coordination to create a multiplicative relationship between a unit of one, a unit of units, and a unit of pairs in activity, which was reflected in the use of multiplicative language like “one times seven is equal to seven”	Established rows and/or columns in arrays as a multiplicative relationship, which enabled them to simultaneously reason about the pairs in a row and column when they were reasoning with multiple pairs

Both MC1 and MC2 students assimilated the situations using two composite units. However, a central difference between MC1 and MC2 students was the extent to which they carried out pairing operations to solve the problems. One way to understand this difference is related to the difference in how MC1 and MC2 students conceive of composite units. MC1 students conceive of composite units as if each unit in a composite unit is different based on the position of the unit in the sequence (Ulrich, 2015). Therefore, it is not surprising that they tended to carry out a pairing operation to produce each pair; if the first unit in a composite unit of three is considered different from the second or third unit in the composite unit of three, then it makes sense that a student would create a pair with each of these units because each one is different. This contrasts from MC2 students who conceive of each of the units of a composite unit as identical (Ulirch). Therefore, once a student made a pair with the first unit of one composite unit, they could operate as if the first unit would be similar to any of the others, concluding what the total number of pairs would be.

Understanding the unit structure that students produced in their solution of Cartesian product problems was a critical component of interpreting how they established relationships between the points in their arrays and the units represented along the axes. In fact, the use of Cartesian product problems was a check on whether and how students had established the points in their arrays as pairs because it provided a context in which to discuss the meaning of points. This issue was well illustrated in Darryl's solution of the Card Problem. In this example, the conversation about the number of face cards in a two-card hand was an occasion for the researcher to realize that although Darryl had initially solved the problem by stating a correct multiplication problem and represented the problem as an array, the points in the array did not represent two-card hands to him. This interpretation suggests that working on Cartesian product problems in concert with developing array representations may have some advantages over other contextual (or decontextualized) situations that involve array representations; the solution of Cartesian product problems can provide a context in which researchers or teachers can investigate students' use of a pairing operation. Working with Cartesian product problems contrasts with situations where a researcher or teacher might ask students to, for example, arrange four rows of six desks in an array-like structure, or four rows of six unit squares into an array-like structure, where there may be no occasion for the student to establish a point in the array as a pair, or a unit square in an array as an area unit composed from length units (Battista, 2007; Nunes, Light, & Mason, 1993; Outhred & Mitchelmore, 2000; see Mulligan & Mitchelmore, 1997 for a study where students were given situations involving rows of desks).

The reverse was also evident, in that questions presented to students about their arrays was a check on the unit structure they had created in solving Cartesian product problems. For example, emergent MC2 students could carry out a minimal number of pairing operations to solve Cartesian product problems (like in Kai's solution to the Vending Machine Problem). Their way of operating was not dissimilar to how Tillema (2013) found MC3 students could operate. However, when emergent MC2 students were presented with problems that involved coordinating the solution of a combinatorics problem with an array, the differences between them and elaborated MC2 or MC3 students were more evident. Jada's solution of the Extension of the Card Problem where she represented the pair at the corner of her array twice was one example of this. This finding supports prior findings that have identified the powerful ways in which having students represent the set of outcomes can support their successful solution of combinatorics problems (Fischbein, Pampu, & Minzat, 1970; Fischbein & Gazit, 1988; Lockwood, 2014; Maher & Yankelewitz, 2010; Outhred, 1996). It builds on these findings by suggesting that this process can also help researchers make key inferences about differences in the nature of the unit structures that students produce. This issue is particularly important for a domain, combinatorics, where students can relatively easily memorize formulas for particular classes of problems without developing important underlying conceptual structures.

6.2 Contributions

One contribution of this paper is that it extends Tillema's (2013) prior research where the goal of this work is to identify mental operations that enable researchers to differentiate between students' multiplicative reasoning on product and isomorphism of measures problems. This contribution is important because it provides an avenue for understanding when students reason about product of measures problems in a way that is consistent with the unique mathematical features that researchers have outlined (e.g., Behr, et. al., 1994; Vergnaud, 1983). Such models are one response to Battista's (2007) call for careful investigations of this issue. Battista made this call in the context of reviewing findings from the literature of students' reasoning about length and area units; however, students are likely to draw on some similar mental operations in relating composite units to pairs in combinatorial contexts (cf. Simon & Blume, 1994).

As part of identifying these mental operations, the findings are also confirmatory of Battista's (2003, 2004) work on how students reason with arrays of squares, in that Battista has found that elementary school students do not initially structure arrays of squares as rows and columns, like the MC1 students in this study. Then students can progress to seeing arrays of squares as rows or columns, but they do not simultaneously coordinate the elements of the rows and columns, which is similar to the emergent MC2 students in this study. Eventually students are able to coordinate the elements of the rows and columns in arrays of squares, like the elaborated MC2 students in this study. This study also elaborates on Battista's work in that the analysis explicitly connects to a framework for investigating the unit structures students create in their multiplicative reasoning (Hackenberg, 2007, 2010; Hackenberg & Tillema, 2009; Steffe, 1992, 1994). It elaborates on both Battista's work and the framework for multiplicative reasoning because neither have explicitly attended to the unique mental operations that students use in constituting product of measures problems.

The findings of this study are consistent with prior usages of Steffe and colleagues' framework for multiplicative reasoning in the following sense. The finding that MC1 students created pairs in activity is aligned with prior findings about MC1 students, in that creating pairs is not dissimilar from creating a unit of units in activity: Creating a unit of units in activity involves creating a group of, for example, four units in activity, and creating a pair is like creating one group of two units in activity. Therefore, the fact that MC1 students were constrained to creating pairs in activity is consistent with prior findings about MC1 students (Hackenberg, 2013; Norton & Wilkins, 2012; Steffe, 1992, 1994; Ulrich, 2015).

A similar observation applies to MC2 students. Emergent MC2 students had interiorized pairs, but did not establish a unit of pairs in activity. This finding is consistent with the fact that emergent MC2 students have interiorized two levels of units, but do not yet work with three levels of units in activity. The reason it is consistent with this finding is that pairs are similar to a unit of units, and a unit of pairs is similar to a unit of units of units structure: A pair is a unit that contains two units and so is like a two-levels-of-unit structure, while a unit of pairs is a unit that contains a multitude of pairs, where each pair contains two units and so is like a three-levels-of-unit structure. This observation also means that the findings about elaborated MC2 students are consistent with prior findings in that some MC2 students can work with three-levels-of-units in activity (e.g., Hackenberg, 2007) because elaborated MC2 students could work with a unit of pairs in activity.

6.3 Implications for Instruction

One implication for instruction from this study is that to leverage the potential power of combinatorics problems teachers need to attend to the kind of unit structures that students create in solving these problems. There are two examples from the data that are most relevant to this implication: Darryl's creation of an array where he did not establish the points as pairs, and Jada's relative ease in identifying a row and column in her array without establishing a row or column as a multiplicative relationship (i.e., as one unit times seven units equal to seven pairs). In both cases, the students operated in combinatorial situations in a way that made them look like they had established a more complex unit structure than they actually had. A way for teachers to act on this implication is through questioning that focuses on the relationships between array representations and the combinatorial situations that they represent. This kind of questioning in both instances revealed that students had not created the kind of unit structure that may have first been indicated by their solution of the problem.

A second important implication for instruction that follows from the first is that representation of combinatorics problems as arrays supported student-teacher communication about the problems. Other researchers have highlighted the important role that representing the set of outcomes can have in students' successfully solving combinatorics problems (Fischbein & Gazit, 1988; Lockwood, 2014; Maher, Powell, & Uptegrove, 2010). This study supports the notion that representing the set of outcomes can also play a clarifying role for teachers about what students "see" in a situation.

7. Conclusion

The second-order models of how MC1, emergent MC2, and elaborated MC2 students reason about combinatorics problems and arrays are intended to be orienting not deterministic (Steffe & Thompson, 2000). That is, the models can help to orient a teacher or researcher to the kinds of interactions that might be profitable for students operating with a particular multiplicative concept; they are not intended to pre-determine the outcome of these interactions. In fact, second-order models are intended to be instruments of interaction (Ulrich, Tillema, Hackenberg & Norton, 2014). As such, one future direction for research is to investigate the learning that students using different multiplicative concepts engage in over time in the solution of combinatorics problems. This work will help to further map productive possibilities as well as possible constraints that a teacher or researcher might experience with students. A second direction for future research is to investigate how students' solutions of combinatorics problems can help them to develop non-linear meanings of multiplication. This direction for further research is made possible because of the explicit attention to how students establish relationships between the units on the axes of an array and the pairs contained in the array; one aspect of students establishing non-linear meanings of multiplication entails establishing explicit relationships about how changes in the number of units on the axes produce related changes in the number of pairs (e.g., understanding that doubling the shirts and pants in the Outfits Problem produces four times the number of outfits). Both directions provide important opportunities for further work in this area.

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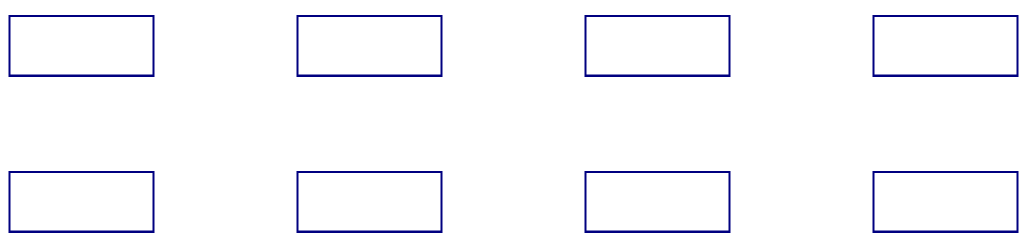
Appendix A: Selection Interview Tasks

1)
   a. A 7th grade classroom has 4 rows of desks with 14 desks in each row. How many desks are in the classroom?

   b. Suppose that 28 desks are added to the classroom. How many rows and desk would there be? Now 1 row is added to the classroom. How many rows and desks would there be?

2) You and 11 friends go to the movies (12 people total). You and your friends fill the first 3 rows of the theater. Determine the number of seats in the theater if there are 13 rows. (Again all rows have the same number of seats.)

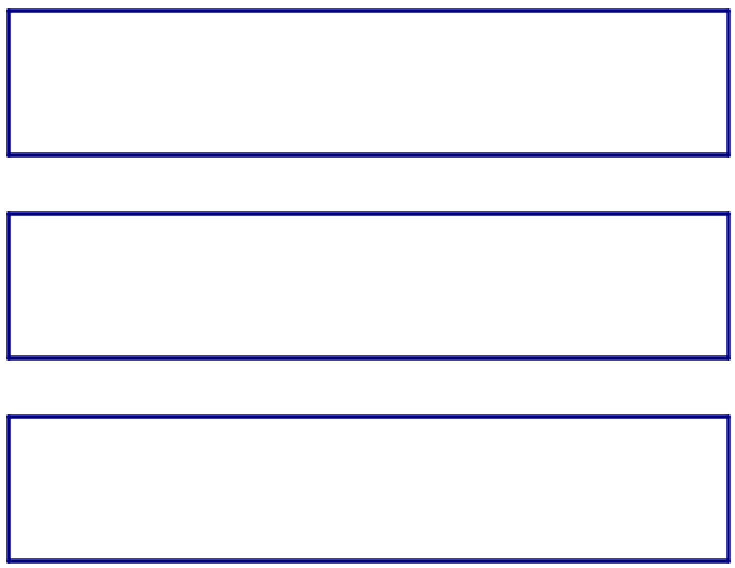
3) The 8th grade is taking 6 buses on a field trip. There are 96 students in the 8th grade. Use the picture below to show how many students will be on each bus (assume there are an equal number of students on each bus).



4)
   a. A candy factory puts 10 candies in each package and puts 10 packages in each box. The factory currently has 3 boxes, 4 packages, and 7 candies. How many total candies do they have?

   b. You buy 534 candies from the candy factory. How many packages of candy do you get?

5) Five people want to share the three identical candy bars. Use the bars below to show how you would accomplish this goal.



6) Jose buys some rope that is 65 centimeters long. Jose's rope is 5 times the size of the piece of rope you bought. Below is a picture of Jose's rope. Use the picture to help you determine the length of your rope.

7) Stephanie checks out 14 books on spiders from the library to prepare for her science fair project. The number of books Stephanie checked out is 2/7 of the total number of books the library owns on spiders. How many total books does the library own on spiders?

Appendix B: Interview 1 Tasks

1) You have 4 shirts and 3 pairs of pants. An outfit is 1 shirt and 1 pair of pants. Draw a picture to help you determine how many total outfits you can make.

2) Subway has 6 kinds of bread and 7 different kinds of meat. You can make a sandwich by choosing one kind of meat and one kind of bread. Draw a picture to help you determine how many total sandwiches you can order.

3) You have the Ace thru 7 of spades and your friend has the Ace thru 7 of hearts. A two-card hand consists of one card from your friend and one card from you (e.g., the two of spades and the two of hearts would be 1 two-card hand). How many different two-card hands could you make?

4) You and a partner each flip a coin and record the result of each of your coin flips (e.g., tails-tails would mean you each got tails on a flip). How many different outcomes could you record?

5) You are designing a flag for a recently discovered country. The flag has two stripes, each of which can be filled with a color. You have 8 different colors to choose from. Illustrate the total number of flags you could make.

6) Each night at summer camp one camper does the dishes. To select the camper who is going to do the dishes each camper is assigned a two-digit number (e.g., 13 or 02). Balls are then put into a bag with the numbers 0-9 on them and the lead counselor chooses one ball from the bag, he writes this number down, puts the ball back in the bag, and chooses another ball from the bag, writes this number down, which forms a two-digit number. How many campers would the counselor need to make sure the dishes were done each night?

7) A volleyball league has 10 teams. Each team wants to play each other once. How many total games would there be?

8) You are in a room with 12 other people (13 people total). Each person wants to shake all of the other people's hands. How many total handshakes will there be?

9) A circle has 8 points on the circumference. A chord is a segment that can be drawn between 2 different points. How many different chords could you draw on your circle?

10) At a picnic there are 2 kinds of meat, 2 kinds of bread, and 3 kinds of cheese you can use to make a sandwich. A sandwich consists of 1 kind of meat, 1 kind of bread, and 1 kind of cheese. Draw a picture to determine how many different kinds of sandwiches you can make.

Appendix C: Interview 2 Tasks

1) A meal at a local restaurant consists of one salad and one entrée. The restaurant serves 6 different kinds of salad and 14 different kinds of entrées. 10 of the entrees have meat. Illustrate with an array the total number of meals that are vegetarian and the total number of meals that are non-vegetarian.

2) At a Colts game, you can buy 7 kinds of soda and 16 different kinds of sausages. 10 of the sausage are Polish style and the rest are German. Use an array to show the total number of sausage soda combinations you could make. On your array show the combinations that have a Polish style sausage and those that have a German style sausage.

3) You have the ace through king of hearts (13 cards). Your friend has the ace through king of clubs (13 cards). Use an array to show all of the possible 2-card hands you could make that consist of one heart and one club. On your array show the number of 2-card hands that have two face cards (Jack, Queen, King), that have exactly one face card, and that have no face cards. Use the sections of your array to determine the total number of 2-card hands you can make.

4) You have to design a two-character password for your computer. The characters in the password can consist of the letters A thru N. Illustrate with an array all possible passwords. Use the pattern from problem 3 to solve for the total number of passwords.

5) A young woman needs to buy one dress and one pair of shoes for her prom dance. A local store has 17 dresses and 15 pairs of shoes that she can choose from. Use an array to represent the total number of possible prom outfits she can choose from. Suppose that 7 of the dresses are black and 5 of the pairs of shoes are black. Show on your array the total number of prom outfits that would be “all black” (i.e. a black pair of shoes and black dress), the total number of prom outfits that would have “some black” (i.e. either a black pair of shoes or black dress), and the total number of outfits that would “not be black” (i.e. neither black shoes nor black dress). Use these sections of your array to determine how many total prom outfits you can make. Write one multiplication problem that tells you the number of prom outfits you can make.

6) On Valentine’s Day, you buy flowers and a vase for your girlfriend. A local florist has 17 kinds of flowers and 14 vases. 10 kinds of flowers are red and 10 vases are glass. Determine the total number of flower vase combinations there are by determining the number of glass vase and red flower combinations, the number of glass vase and non-red flower combinations, the number of non-glass vase and red flower combinations, and the number of non-glass and non-red flower combinations. Use an array to show your solution.